

Geometric reduction of Hamiltonian systems

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Abstract

Given a foliation \mathcal{S} of a manifold \mathcal{M} , a distribution \mathcal{Z} in \mathcal{M} transversal to \mathcal{S} and a Poisson bivector Π on \mathcal{M} we present a geometric method of reducing this operator on the foliation \mathcal{S} along the distribution \mathcal{Z} . It encompasses the classical ideas of Dirac (Dirac reduction) and more modern theory of J. Marsden and T. Ratiu, but our method leads to formulas that allow for an explicit calculation of the reduced Poisson bracket. Moreover, we analyse the reduction of Hamiltonian systems corresponding to the bivector Π .

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1 Introduction

The reduction theory of dynamical systems consist of two branches: the first branch deals with constrained Lagrangian systems, the second with constrained Hamiltonian systems. In the Lagrangian approach one consider separately the case of holonomic constraints, i.e. the constraints which may depend on velocities, but only in such a way that the equations of constraints can be integrated to eliminate velocities, and the non-holonomic case. In many of these papers one first considers the Lagrangian formulation and then passess to the corresponding Hamiltonian formulation (see for examplpe [1]). The reduction theory in the Hamiltonian context has been initiated by P.A.M. Dirac, who in his famous paper [2] described a method of reducing a given Poisson bracket onto a

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submanifold given by some constraints φ provided they were of "second class". In this approach the classical notion of holonomic constraints is usually not introduced as in this context there is no obvious division of variables between "position" and "velocity" (or "momenta"). Recently, there has been much interest to extend the theory onto the case of generalized Hamiltonian systems (see for example [3]).

The ideas of Dirac were developed in many papers, among others in [4]-[10] (see also the literature quoted there). A geometric meaning of this reduction procedure has been investigated in [8] and in [11].

In this paper we develop the ideas of [2] and [8] and present a constructive, computable method of reducing (locally) a given Poisson operator Π to any regular submanifold \mathcal{S} . The idea of the method is to choose a distribution \mathcal{Z} (not necessarily integrable) that is i) transversal to the foliation \mathcal{S} ii) at any $x \in \mathcal{M}$ it completes $T_x\mathcal{S}$ to $T_x\mathcal{M}$ iii) it makes the operator Π \mathcal{Z} -invariant (see definitions below) and then to deform the Poisson operator Π to a new Poisson operator Π_D such that its image will be tangent to the submanifold \mathcal{S} . This new operator Π_D will be always Poisson (and so its natural restriction to \mathcal{S} will be Poisson). In consequence, we obtain a method of reducing a Hamiltonian system on \mathcal{M} to a Hamiltonian system on every leaf \mathcal{S}_ν of the foliation \mathcal{S} . This reduced system strongly depends on the choice of the distribution \mathcal{Z} . As a special case we obtain the classical Dirac reduction of Hamiltonian system. All our considerations will be local in the sense that our manifold \mathcal{M} is perhaps only an open submanifold of a larger manifold. Our construction is equivalent to the reduction method proposed by Marsden and Ratiu in [8]. However, our approach has advantages: it can be performed simultaneously on any leaf \mathcal{S}_ν of the foliation \mathcal{S} , it is constructive (the approach of Marsden and Ratiu requires calculations of prolongations of \mathcal{Z} -invariant functions and as such is difficult to perform in practice) and it is formulated in the language of Poisson bivectors rather than Poisson brackets.

We want to stress that this method does not require the submanifold to be given by a holonomic constraints on some configuration space. In fact, we do not require our manifold to be a cotangent bundle to any configurational manifold at all, but of course our construction covers this special case as well. For example, it covers the case discussed in [1], where the authors obtain the Poisson operator on the constrained submanifold only in the case of holonomic constraints - because they simply restrict their Poisson operator Π onto the constrained submanifold, and such restriction usually (apart from the holonomic case) destroys the poissonity of the operator.

We have to notice that some particular versions of the proposed scheme appeared recently in [12] and [13]. In [14] the author applied the same setting as above but with no conclusive formulas for computing the actual deformed Poisson bracket Π_D and since he did not use the notion of \mathcal{Z} -invariance, his deformed operator Π_D , although formally identical with our construction, was not always Poisson (it has been called there a pseudo-Poisson operator).

Some basic steps of our construction have been presented in our previous paper [15]. This paper, however, was mainly devoted to completion of the

above picture by its "dual" part by developing a theory of a Marsden-Ratiu type reduction of presymplectic 2-forms Ω that are (in a sense) dual to a given Poisson operator Π . The picture presented here is much clearer and moreover it is parametrization-free in the sense that we prove the main results without necessity of discussing some particular functions defining our foliation \mathcal{S} . Moreover, in the paper [15] we focused on Dirac case while this paper has a general character. In the end, in this paper we also consider the related reduction of Hamiltonian dynamics.

2 Geometric reduction of Poisson bivectors

Let us consider a smooth manifold \mathcal{M} of arbitrary (finite) dimension n and a foliation \mathcal{S} of \mathcal{M} consisting of the leaves \mathcal{S}_ν parametrized by $\nu \in \mathbf{R}^k$ (so that $k \in \mathbf{N}$ is the codimension of every leaf \mathcal{S}_ν). Consider also a regular distribution \mathcal{Z} on \mathcal{M} (that is a smooth collection of the spaces $\mathcal{Z}_x \subset T_x\mathcal{M}$ where ν is such that $x \in \mathcal{S}_\nu$) such that it completes every $T_x\mathcal{S}$ to $T_x\mathcal{M}$ in the sense that

$$T_x\mathcal{M} = T_x\mathcal{S}_\nu \oplus \mathcal{Z}_x \quad (1)$$

for every x in \mathcal{M} . Here and in what follows \oplus denotes the direct sum of vector spaces. It means that every vector field X on the manifold \mathcal{M} has a unique decomposition $X = X_\parallel + X_\perp$ such that for every x in \mathcal{M} the vector $(X_\parallel)_x \in T_x\mathcal{S}$ (X_\parallel is tangent to the leaves of the foliation \mathcal{S}) while $(X_\perp)_x \in \mathcal{Z}_x$ (X_\perp is contained in the distribution \mathcal{Z}). The splitting (1) induces the following splitting of the corresponding dual space $T_x^*\mathcal{M}$:

$$T_x^*\mathcal{M} = T_x^*\mathcal{S}_\nu \oplus \mathcal{Z}_x^* \quad (2)$$

where $T_x^*\mathcal{S}_\nu$ is the annihilator of \mathcal{Z}_x while the space \mathcal{Z}_x^* is the annihilator of $T_x\mathcal{S}_\nu$. Thus, any one-form α on \mathcal{M} has a unique decomposition $\alpha = \alpha_\parallel + \alpha_\perp$ such that $(\alpha_\parallel)_x \in T_x^*\mathcal{S}$ (α_\parallel annihilates the vectors from \mathcal{Z}) while $(\alpha_\perp)_x \in \mathcal{Z}_x^*$ (α_\perp annihilates the vectors tangent to the foliation \mathcal{S}). We will call X_\parallel and α_\parallel as projections of X and α (respectively) on the foliation \mathcal{S} . Abusing notation a bit we will write that $X \subset T\mathcal{S}$ if $X = X_\parallel$, $X \subset \mathcal{Z}$ if $X = X_\perp$ and similarly for one forms: $\alpha \subset T^*\mathcal{S}$ if $\alpha = \alpha_\parallel$, $\alpha \subset \mathcal{Z}^*$ if $\alpha = \alpha_\perp$.

Let us now suppose that our manifold \mathcal{M} is equipped with a Poisson bivector Π (i.e. a bivector with vanishing Schouten bracket, see [6]). This operator induces the Poisson bracket $\{F, G\}_\Pi = \langle dF, \Pi dG \rangle$ on the algebra of smooth functions on \mathcal{M} , where $\langle \cdot, \cdot \rangle$ is the dual map between $T^*\mathcal{M}$ and $T\mathcal{M}$. A smooth real-valued function F on \mathcal{M} is called \mathcal{Z} -invariant if the Lie derivative $L_Z F = 0$ for any vector field $Z \subset \mathcal{Z}$. We will now adopt the following definition.

Definition 1 *The operator Π is said to be \mathcal{Z} -invariant if $L_Z \{F, G\}_\Pi = 0$ for any pair of \mathcal{Z} -invariant functions F and G and every vector field $Z \subset \mathcal{Z}$.*

Notice that our definition does not necessarily mean that $L_Z \Pi = 0$ for all vector fields $Z \subset \mathcal{Z}$, as for any pair F, G of \mathcal{Z} -invariant functions the condition

$L_Z \{F, G\}_{\Pi} = 0$ means only that the function $\langle dF, (L_Z \Pi) dG \rangle$ vanishes. Thus, Π does not have to be an invariant of the distribution \mathcal{Z} to be \mathcal{Z} -invariant in our meaning. Notice also, that the above definition is equivalent to the statement that for any pair $\alpha, \beta \in T^* \mathcal{S}$ we have $\langle \alpha, (L_Z \Pi) \beta \rangle = 0$ (since if F is \mathcal{Z} -invariant then $dF \in T^* \mathcal{S}$).

Suppose for the moment that the distribution \mathcal{Z} is spanned by k vector fields Z_i . We say, that the operator Π is *Vaisman* [16] with respect to \mathcal{Z} if for every vector field Z_i there exists vector fields W_{ij} , $j = 1, \dots, k$ such that

$$L_{Z_i} \Pi = \sum_{j=1}^k W_{ij} \wedge Z_j. \quad (3)$$

It is easy to see that this definition does not depend on the choice of basis in \mathcal{Z} (although the vector fields W_{ij} obviously do). If the operator Π is Vaisman with respect to \mathcal{Z} , then it is also \mathcal{Z} -invariant, as then for any two one-forms $\alpha, \beta \in T^* \mathcal{S}$

$$\langle \alpha, (L_Z \Pi) \beta \rangle = \sum_{j=1}^k \langle \alpha, (W_{ij} \wedge Z_j) \beta \rangle = 0,$$

since α and β annihilate all the vector fields Z_i . The converse statement is however not true in general.

Let us now consider a Poisson operator Π on \mathcal{M} and define the following bivector:

$$\Pi_D (\alpha, \beta) = \Pi(\alpha_{||}, \beta_{||}) \quad \text{for any pair } \alpha, \beta \text{ of one-forms.} \quad (4)$$

We will often call the bivector Π_D a *deformation* of Π . This bivector occurred for example in [12] in a more restrictive context. Observe that it always exists and that it is uniquely defined once the foliation \mathcal{S} and the distribution \mathcal{Z} are given (it is thus a purely geometric construction). It has an important property: its image lies always in $T\mathcal{S}$.

Lemma 2 $\Pi_D(\alpha) \in T\mathcal{S}$ for any one-form α on \mathcal{M} , i.e. the image of Π_D is tangent to the foliation \mathcal{S} .

Proof. We have to show that $\langle \beta, \Pi_D \alpha \rangle = 0$ for any $\beta \in Z^*$. But

$$\langle \beta, \Pi_D \alpha \rangle = \Pi_D (\beta, \alpha) = \Pi (\beta_{||}, \alpha_{||}) = 0$$

since $\beta_{||} = 0$ for every $\beta \in Z^*$. ■

Thus, the deformed bivector Π_D has its image in $T\mathcal{S}$ and if we consider it as mapping from one-forms to vector fields on \mathcal{M} then it can be naturally restricted to a bivector π_{R_ν} on every leaf \mathcal{S}_ν of \mathcal{S} by simply restricting its domain to \mathcal{S}_ν :

$$\pi_{R_\nu} = \Pi_D|_{\mathcal{S}_\nu}.$$

Moreover, it induces a new bracket for functions on \mathcal{M} :

$$\{F, G\}_{\Pi_D} = \Pi_D (dF, dG) = \Pi((dF)_{||}, (dG)_{||}).$$

Of course, the bivector Π_D (and thus even π_{R_ν}) in general does not have to be Poisson. However, it turns out that if Π is \mathcal{Z} -invariant then Π_D (and thus every π_{R_ν}) is Poisson.

Theorem 3 *If Π is \mathcal{Z} -invariant then Π_D given by (4) is Poisson.*

Proof. We will prove that the bracket $\{\cdot, \cdot\}_{\Pi_D}$ is Poisson. Obviously, this bracket is antisymmetric and satisfies the Leibniz property. It remains to show, that it also satisfies the Jacobi identity, that is $\{\{F, G\}_{\Pi_D}, H\}_{\Pi_D} + \text{cycl.} = 0$, for any functions F, G, H . Using the definition of Π_D , this condition can be written as

$$\left\langle d \left\langle (dF)_\parallel, \Pi(dG)_\parallel \right\rangle, \Pi(dH)_\parallel \right\rangle + \text{cycl.} = 0. \quad (5)$$

However, for any vector field $Z \in \mathcal{Z}$ we have

$$\left\langle d \left\langle (dF)_\parallel, \Pi(dG)_\parallel \right\rangle, Z \right\rangle = Z \left(\left\langle (dF)_\parallel, \Pi(dG)_\parallel \right\rangle \right) = L_Z \left\langle (dF)_\parallel, \Pi(dG)_\parallel \right\rangle = 0$$

due to the assumed \mathcal{Z} -invariance of Π . This means that $d \left\langle (dF)_\parallel, \Pi(dG)_\parallel \right\rangle \subset T^*\mathcal{S}$, so that

$$\left(d \left\langle (dF)_\parallel, \Pi(dG)_\parallel \right\rangle \right)_\parallel = d \left\langle (dF)_\parallel, \Pi(dG)_\parallel \right\rangle,$$

and thus condition (5) turns out to be the Jacobi identity for Π , which is satisfied since Π is Poisson. ■

Thus, given a foliation \mathcal{S} on \mathcal{M} and a transversal distribution \mathcal{Z} on \mathcal{M} (such that (1) is satisfied) we can reduce any Poisson bivector Π that is \mathcal{Z} -invariant to a Poisson bivector π_{R_ν} on the leaf \mathcal{S}_ν of \mathcal{S} by deforming Π to Π_D and then by restricting Π_D to \mathcal{S}_ν . This construction yields the same operator π_{R_ν} as in the approach of Marsden and Ratiu [8]. We will however show how this construction can be easily realized in practice.

Remark 4 *In case when the foliation \mathcal{S} coincides with the symplectic foliation of Π we have of course that $\Pi_D = \Pi$, since in this case $\Pi((dF)_\perp) = 0$ for any function F and so $\{F, G\}_{\Pi_D} = \Pi((dF)_\parallel, (dG)_\parallel) = \Pi(dF, dG) = \{F, G\}_\Pi$. In this case π_{R_ν} is the standard reduction of Π on its symplectic leaf \mathcal{S}_ν .*

Let us now consider some special cases of our general situation. We firstly observe that the annihilator \mathcal{Z}^* of $T\mathcal{S}$ is defined as soon as the foliation \mathcal{S} is determined (we do not need to specify a particular \mathcal{Z} in order to define \mathcal{Z}^*).

Definition 5 *The distribution $\mathcal{D} = \Pi(\mathcal{Z}^*)$ (so that $\mathcal{D}_x = \Pi(\mathcal{Z}_x^*)$) is called a Dirac distribution associated with the foliation \mathcal{S} .*

Thus, the distribution \mathcal{D} is determined by \mathcal{S} and by Π . A priori, two limit cases are possible here. If $T\mathcal{M} = \mathcal{D} \oplus T\mathcal{S}$ we say that we are in the *Dirac case*. If $\mathcal{D} \subset T\mathcal{S}$ we say that we are in the *tangent case*. In the Dirac case we have a canonical choice of \mathcal{Z} : we can choose $\mathcal{Z} = \mathcal{D}$ (in this case Π is automatically \mathcal{Z} -invariant, since \mathcal{Z} is spanned by the vector fields Hamiltonian with respect to Π). Nevertheless, we can also choose some other distribution $\mathcal{Z} \neq \mathcal{D}$. In the tangent case we have no canonical choice of \mathcal{Z} and we are free to find a distribution \mathcal{Z} that makes Π \mathcal{Z} -invariant. Anyway, in both cases we have many non-equivalent deformations Π_D (and thus projections π_{R_ν}). Generically, the distribution \mathcal{D} will not be tangent to \mathcal{S} , but it will not suffice to span $T\mathcal{M}$ together with $T\mathcal{S}$.

Let us now suppose that the foliation \mathcal{S} of \mathcal{M} is parametrized by the set of k functionally independent real valued functions $\varphi_i(x)$ so that its leaves have the form $\mathcal{S}_\nu = \{x \in \mathcal{M} : \varphi_i(x) = \nu_i, \nu_i \in \mathbf{R}, i = 1, \dots, k\}$ where k is - as above - the codimension of the foliation. We will show how the above considerations can be written in the parametrization $\{\varphi_i\}$. The one-forms $d\varphi_i$ constitute a basis in \mathcal{Z}^* . Then, the Dirac distribution \mathcal{D} is spanned by k (possibly dependent) Hamiltonian vector fields $X_i = \Pi d\varphi_i$. Let us denote a basis of \mathcal{Z} dual to the basis $\{d\varphi_i\}$ in \mathcal{Z}^* by Z_i , so that $Z_i(\varphi_j) = \delta_{ij}$. Our projections $X_{||}$ and $\alpha_{||}$ are then given by

$$X_{||} = X - \sum_{i=1}^k X(\varphi_i) Z_i,$$

(obviously $X_{||}(\varphi_i) = 0$ for all i so that indeed this vector field is tangent to the leaves of \mathcal{S}) and by

$$\alpha_{||} = \alpha - \sum_{i=1}^k \alpha(Z_i) d\varphi_i$$

(and obviously $\alpha_{||}(Z_i) = 0$ for all i). Thus

$$\begin{aligned} \Pi(\alpha_{||}, \beta_{||}) &= \Pi \left(\alpha - \sum_{i=1}^k \alpha(Z_i) d\varphi_i, \beta - \sum_{j=1}^k \beta(Z_j) d\varphi_j \right) = \\ &= \Pi(\alpha, \beta) - \sum_{j=1}^k \beta(Z_j) \Pi(\alpha, d\varphi_j) - \sum_{i=1}^k \alpha(Z_i) \Pi(d\varphi_i, \beta) + \\ &\quad + \sum_{i,j=1}^k \alpha(Z_i) \beta(Z_j) \Pi(d\varphi_i, d\varphi_j), \end{aligned}$$

so that the deformation Π_D can be expressed as

$$\Pi_D = \Pi - \sum_i X_i \wedge Z_i + \frac{1}{2} \sum_{i,j} \varphi_{ij} Z_i \wedge Z_j \quad (6)$$

where the functions φ_{ij} are defined as

$$\varphi_{ij} = \{\varphi_i, \varphi_j\}_\Pi = X_j(\varphi_i) = \Pi(d\varphi_i, d\varphi_j).$$

In the Dirac case all the vector fields X_i are transversal to the foliation \mathcal{S} and are moreover linearly independent. It happens precisely when $\det(\varphi_{ij}) \neq 0$ (the

functions φ_i are then 'second class constraints' in the terminology of Dirac). The vector fields Z_i (the dual basis to $\{d\varphi_i\}$) can be expressed through the vector fields X_i as

$$Z_i = \sum_{j=1}^k (\varphi^{-1})_{ji} X_j, \quad i = 1, \dots, k$$

Indeed,

$$Z_j(\varphi_i) = \sum_{s=1}^k (\varphi^{-1})_{sj} X_s(\varphi_i) = \sum_{s=1}^k (\varphi^{-1})_{sj} \varphi_{is} = \delta_{ij}.$$

Moreover, in this case the deformation (6) attains the form:

$$\Pi_D = \Pi - \frac{1}{2} \sum_{i=1}^k X_i \wedge Z_i. \quad (7)$$

This operator defines the following bracket on $C^\infty(\mathcal{M})$

$$\{F, G\}_{\Pi_D} = \{F, G\}_\Pi - \sum_{i,j=1}^k \{F, \varphi_i\}_\Pi (\varphi^{-1})_{ij} \{\varphi_j, G\}_\Pi, \quad (8)$$

which is just the well known *Dirac deformation* [2] of the bracket $\{.,.\}_\Pi$.

In the tangent case all the vector fields X_i are tangent to the foliation \mathcal{S} and the deformation (6) attains the form

$$\Pi_D = \Pi - \sum_{i=1}^k X_i \wedge Z_i, \quad (9)$$

and has been considered in [12] and in [13].

In [11] we have considered a bit more general then (6) deformation of Π of the form $\Pi_d = \Pi - \sum_i V_i \wedge Z_i$. The image of Π_d does not have to lie in $T\mathcal{S}$. However, one can prove that it happens precisely when the vector fields V_i satisfy the following functional equation

$$V_i = X_i + \sum_j V_j(\varphi_i) Z_j. \quad (10)$$

and in this case Π_d is Poisson. Substituting this formula into Π_d yields

$$\Pi_d = \Pi - \sum_{i=1}^k X_i \wedge Z_i + \sum_{i,j=1}^k V_j(\varphi_i) Z_i \wedge Z_j, \quad (11)$$

and it can be proved (after some technical manipulations) that in this case $\Pi_d = \Pi_D$ (even though $V_j(\varphi_i)$ does not have to be equal to $\frac{1}{2}\varphi_{ij}$). A natural solution of the equation (10) in the Dirac case is given by $V_i = \frac{1}{2}X_i$ and in the tangent case by $V_i = X_i$, which turns (11) into the deformations (7) and (9) respectively. Thus, in a sense, the deformation Π_D is the canonical deformation in our setting.

Our construction is strongly related with the construction of Marsden and Ratiu. J. Marsden and T. Ratiu presented in [8] a natural way of reducing of a given Poisson bracket $\{\cdot, \cdot\}_\Pi$ on \mathcal{M} to a Poisson bracket $\{\cdot, \cdot\}_{\pi_R}$ on a given submanifold \mathcal{S}_ν (in our notation). Their method is non-constructive in the sense that in order to find the bracket $\{f, g\}_{\pi_R}$ of two functions $f, g : \mathcal{S}_\nu \rightarrow R$ one has to calculate $\mathcal{Z}|_{\mathcal{S}_\nu}$ -invariant prolongations of these functions. Our construction is performed on the level of bivectors rather than on the level of Poisson brackets. This construction (by deformation of the bivector Π) applies directly to every leaf of the distribution \mathcal{S} and moreover it is constructive. At every leaf, however, both constructions are equivalent. On the other hand, we make the assumption about the transversality of the distribution \mathcal{Z} that was not present in the original paper of Marsden and Ratiu. This assumption is however very natural since it makes all the assumptions of Poisson Reduction Theorem in [8] automatically satisfied.

3 Reduction of Hamiltonian dynamics

Let us begin this section by stating - in our setting - a well-known theorem about the relation between the Dirac deformation Π_D of Π and the dynamic imposed by the constraints. Suppose, thus, that our manifold \mathcal{M} is a cotangent bundle of a Riemannian manifold \mathcal{Q} with a covariant metric tensor g , so that $\mathcal{M} = T^*\mathcal{Q}$. Denote the corresponding contravariant metric tensor by G . Consider a Lagrangian dynamical system on $T\mathcal{Q}$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, \dots, n, \quad (12)$$

with a potential Lagrangian function $L(q, \dot{q}) = \frac{1}{2} \dot{q}^t g \dot{q} - V(q)$. This of course leads to a Hamiltonian equations of motion on $\mathcal{M} = T^*\mathcal{Q}$

$$\dot{q}_i = \{q_i, H\}_\Pi, \quad \dot{p}_i = \{p_i, H\}_\Pi \quad (13)$$

with the Hamiltonian $H = \frac{1}{2} p^t G p + V(q)$ and with the canonical Poisson operator Π . Let us now impose a physical constraint $\varphi(q) = 0$ on our system and assume that in the beginning the coordinates of the system lie on the submanifold \mathcal{Q}_0 of \mathcal{Q} given by $\varphi(q) = 0$. One often makes a physical assumption here that the surface \mathcal{Q}_0 starts to act on our system with a reaction force $R(q, \dot{q})$ that is orthogonal to \mathcal{Q}_0 and such that the trajectories of the constrained system

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = R_i(q, \dot{q}) \quad (14)$$

that start on \mathcal{Q}_0 remain on \mathcal{Q}_0 . On the level of the phase space $\mathcal{M} = T^*\mathcal{Q}$ the Hamiltonian system (13) is now subordinate to a pair of constraints:

$$\varphi_1(q) \equiv \varphi(q) = 0, \quad \varphi_2(q) \equiv (\nabla \varphi)^t G p = 0,$$

where $\nabla\varphi$ is the gradient of φ with respect to q -variables (differential of φ in \mathcal{Q}), (the second constraint is a consequence of the fact that the velocities \dot{q} must remain tangent to \mathcal{Q}_0) and thus modifies to

$$\dot{q}_i = \{q_i, H\}_{\Pi} \quad , \quad \dot{p}_i = \{p_i, H\}_{\Pi} + R_i(q, p). \quad (15)$$

Theorem 6 *The equations (15) are Hamiltonian and can be written in the form*

$$\dot{q}_i = \{q_i, H\}_{\Pi_D} \quad , \quad \dot{p}_i = \{p_i, H\}_{\Pi_D} \quad (16)$$

where $\Pi_D = \Pi - \frac{1}{2} \sum_{i=1}^2 X_i \wedge Z_i$ is the Dirac deformation (7) of Π given by the constraints φ_1 and φ_2 .

Thus, the response of the Lagrangian system (12) subordinated to the reaction forces R can be accounted for by the corresponding Dirac deformation of the Poisson operator Π . We will only sketch the proof.

Proof. The reaction force R can be calculated by differentiating the assumed identity $\varphi(q(t)) \equiv 0$ twice with respect to t and eliminating the second derivatives \ddot{q}_i with the help of equations (14) and by using the demand that the force should be orthogonal to \mathcal{Q}_0 . After some calculations we obtain that:

$$R(q, p) = \frac{1}{(\nabla\varphi)^t G \nabla\varphi} \left((\nabla\varphi)^t G \nabla V - (p^t G) H_{\varphi}(Gp) + A \right) \nabla\varphi \quad (17)$$

where H_{φ} is the Hessian of φ : $(H_{\varphi})_{ij} = \frac{\partial^2 \varphi}{\partial q_i \partial q_j}$ and where $A = A(q, p)$ is given by

$$A = \sum_s \frac{\partial \varphi}{\partial q_s} \sum_{r, m, i, j} \Gamma_{ij}^s G^{ir} G^{jm} p_r p_m$$

so that it vanishes in the Euclidean coordinates when all Christoffel's symbols Γ_{jk}^i are equal to zero. On the other hand, calculating the explicit form of (16) on the submanifold of $\mathcal{M} = T^*\mathcal{Q}$ given by the constraints φ_1, φ_2 leads to the equations (15) with R given by (17). ■

Let us now consider a Hamiltonian vector field $X = \Pi dH$ on a general Poisson manifold \mathcal{M} where H is some real-valued smooth function on \mathcal{M} (Hamiltonian function). We constantly assume that we have a smooth, regular foliation \mathcal{S} on \mathcal{M} and a regular distribution \mathcal{Z} on \mathcal{M} such that (1) is satisfied. The corresponding Π_D defined by (4) is Poisson and has its image tangent to the foliation \mathcal{S} , so that it can be properly restricted on every leaf \mathcal{S}_{ν} of \mathcal{S} . Thus, the following definition makes sense.

Definition 7 *We call the vector field $X_D = \Pi_D dH$ the Hamiltonian projection of the Hamiltonian vector field $X = \Pi dH$.*

The vector field X_D lives on every leaf of the foliation in the sense that its restriction on the leaf \mathcal{S}_{ν} is tangent to \mathcal{S}_{ν} . Moreover, on the leaf \mathcal{S}_{ν} it coincides with the Hamiltonian vector field $\pi_{R_{\nu}} dh$:

$$\Pi_D dH|_{\mathcal{S}_{\nu}} = \pi_{R_{\nu}} dh,$$

where $h = H|_{\mathcal{S}_\nu}$ is the restriction of the Hamiltonian H to the leaf \mathcal{S}_ν . To see this it is enough to choose a parametrization $\{\varphi_i\}$ of \mathcal{S} and to pass to any system of coordinates of the form (x, φ) . In these coordinates the bivector Π_D has a matrix form with a non-zero upper-left block coinciding with the matrix form of π_{R_ν} and with the remaining terms equal to zero.

There is a connection between $X = \Pi dH$ and its Hamiltonian projection $X_D = \Pi_D dH$.

Theorem 8 (*Dynamic reduction theorem*) *If $X = \Pi dH$, $X_D = \Pi_D dH$ and $X_i = \Pi d\varphi_i$ then*

$$X_D = X_{\parallel} - \sum_{i=1}^k Z_i(H) X_{i,\parallel}. \quad (18)$$

Proof. A direct calculation yields

$$\begin{aligned} X_D = \Pi_D dH &= X - \sum_{i=1}^k (Z_i(H) X_i - X_i(H) Z_i) + \sum_{i,j=1}^k \varphi_{ij} Z_j(H) Z_i = \\ &= X_{\parallel} - \sum_{i=1}^k Z_i(H) \left(X_i - \sum_{j=1}^k \varphi_{ji} Z_j \right), \end{aligned}$$

the last equality due to $X_i(H) = \langle dH, \Pi d\varphi_i \rangle = -\langle d\varphi_i, \Pi dH \rangle = -X(\varphi_i)$. Since $\varphi_{ji} = X_i(\varphi_j)$ it yields (18). ■

Observe that the difference between X_D and X_{\parallel} is the term $\sum_{i=1}^k Z_i(H) X_{i,\parallel}$ that is tangent to the foliation \mathcal{S} , as it should be.

Since for the Dirac case $X_{i,\parallel} = 0$ (by definition) we have

Corollary 9 *In the Dirac case $X_D = X_{\parallel}$, so that in this case the Hamiltonian projection is just the natural projection (in the sense of direct sum) along the distribution \mathcal{Z} .*

The term X_{\parallel} in X_D has a well-known physical interpretation: it describes the evolution of the system $X = \Pi dH$ imposed with the constraints given by φ_i . The physical meaning of the second term in X_D is not clear for us, although it should represent some additional force (friction) acting on the system and tangent to the constraints. The authors are grateful for any hints in this matter.

We are now in position to discuss the degeneracy of Π_D using the above Dynamic reduction theorem (Theorem 8). Let us first discuss the Dirac case.

Proposition 10 *Consider the Dirac deformation Π_D given by (7) of a Poisson operator Π on M . Suppose that the real valued functions c_i , $i = 1, \dots, s$ on M are such that they span the kernel of the operator Π (i.e. are Casimir functions of Π in the sense that $\Pi dc_i = 0$) and such that the functions $\{c_i, \varphi_j\}$ constitute a functionally independent set. Then*

- i) *the constraints φ_i and the 'old' Casimirs c_i are all Casimirs of Π_D*
- ii) *any Casimir of Π_D must be of the form $C(c_1, \dots, c_s, \varphi_1, \dots, \varphi_k)$.*

Proof. The proof of i) is just a calculation. To prove ii), let us complete the functionally independent functions $\{c_i, \varphi_j\}$ to a coordinate system $\{c_i, \varphi_j, x_k\}$ on \mathcal{M} . Suppose that a function $C = C(c, \varphi, x)$ is a Casimir of Π_D given by (7), i.e. that $\Pi_D dC = 0$. Then, according to Theorem 8, $(\Pi dC)_\parallel = 0$, i.e. $\Pi dC \subset \mathcal{Z}$. In the Dirac case the distribution \mathcal{Z} is spanned by the vector fields X_i so that there must exist functions α_i such that $\Pi dC = \sum_{i=1}^k \alpha_i X_i = \sum_{i=1}^k \alpha_i \Pi d\varphi_i = \Pi \left(\sum_{i=1}^k \alpha_i d\varphi_i \right)$. Thus, $\Pi \left(dC - \sum_{i=1}^k \alpha_i d\varphi_i \right) = 0$ or $dC = \sum_{i=1}^k \alpha_i d\varphi_i + \sum_{i=1}^s \beta_i dc_i$ which proves ii). ■

Thus, we can state that the Dirac deformation (7) preserves all the old Casimir functions and introduces new Casimirs φ_i and that no other Casimirs arise in this process. The situation in the general case is more complicated, since the Casimirs of Π does not have to survive the general deformation (4) (or (6)) and moreover since new Casimirs, different from φ_i ones, can arise. We can merely state that in the general case the function C is a Casimir of Π_D if and only if the vector field $Y = \Pi dC$ satisfies the relation

$$Y_\parallel = \sum_{i=1}^k Z_i(C) X_{i,\parallel}$$

which for the Dirac case degenerates to the already discussed condition $Y_\parallel = 0$.

4 Example

Let us conclude this article by an example. Consider the so called first Newton representation of the seventh-order stationary flow of the KdV hierarchy [18],[19],[17]. It is the following system of second order Newton equations

$$\begin{aligned} q_{1,tt} &= -10q_1^2 + 4q_2 \\ q_{2,tt} &= -16q_1q_2 + 10q_1^3 + 4q_3 \\ q_{3,tt} &= -20q_1q_3 - 8q_2^2 + 30q_1^2q_2 - 15q_1^4 \end{aligned}$$

(where the subscript t denotes the differentiation with respect to the evolution parameter t). By putting $p_1 = q_{3,t}$, $p_2 = q_{2,t}$, $p_3 = q_{1,t}$ it can be written in a Hamiltonian form:

$$\frac{d}{dt} (q_1, q_2, q_3, p_1, p_2, p_3)^T = X = \Pi dH$$

where Π is the canonical Poisson operator on the space $\mathcal{M} = \{(q_1, q_2, q_3, p_1, p_2, p_3)\}$,

$$\Pi = \left[\begin{array}{c|c} 0 & I_3 \\ \hline -I_3 & 0 \end{array} \right]$$

and with the Hamiltonian

$$H = p_1p_3 + \frac{1}{2}p_2^2 + 10q_1^2q_3 - 4q_2q_3 + 8q_1q_2^2 - 10q_1^3q_2 + 3q_1^5.$$

Consider also a foliation \mathcal{S} given by a pair of constraints

$$\varphi_1 = q_3 + q_1 q_2, \quad \varphi_2 = p_1 + q_1 p_2 + q_2 p_3.$$

where φ_2 is the so called G -consequence of φ_1 (i.e. a lift from the configuration space $\{(q_1, q_2, q_3)\}$ to \mathcal{M}) with respect to the antidiagonal metric tensor G [17]. The vector fields $X_i = \Pi d\varphi_i$ have the form

$$X_1 = -q_2 \frac{\partial}{\partial p_1} - q_1 \frac{\partial}{\partial p_2} - \frac{\partial}{\partial p_3}, \quad X_2 = \frac{\partial}{\partial q_1} + q_1 \frac{\partial}{\partial q_2} + q_2 \frac{\partial}{\partial q_3} - p_2 \frac{\partial}{\partial p_1} - p_3 \frac{\partial}{\partial p_2}$$

and they are transversal to \mathcal{S} so that we have the Dirac case. Thus, the distribution $\mathcal{Z} = \mathcal{D} = Sp\{X_i\}$ makes Π \mathcal{Z} -invariant. The basis in \mathcal{Z} that is dual to $\{d\varphi_i\}$ is $Z_1 = \frac{1}{\varphi_{12}} X_2$, $Z_2 = -\frac{1}{\varphi_{12}} X_1$, where $\varphi_{12} = \{\varphi_1, \varphi_2\}_\Pi = 2q_2 + q_1^2$. The Dirac deformation Π_D given by (7) attains in the adapted coordinate system $\{(q_1, q_2, \varphi_1, \varphi_2, p_2, p_3)\}$ the form

$$\Pi_D = \frac{1}{2q_2 + q_1^2} \begin{bmatrix} 0 & 0 & 0 & 0 & -q_1 & -1 \\ 0 & 0 & 0 & 0 & 2q_2 & -q_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ q_1 & -2q_2 & 0 & 0 & 0 & p_3 \\ 1 & q_1 & 0 & 0 & -p_3 & 0 \end{bmatrix}$$

It has, as it should, two Casimirs φ_1, φ_2 . We can now easily restrict Π_D to the operator π_{R_ν} on \mathcal{S}_ν . If we parametrize \mathcal{S}_ν with the coordinates $\{(q_1, q_2, p_2, p_3)\}$ (the constraints φ_1, φ_2 are constant on every \mathcal{S}_ν) then

$$\pi_{R_\nu} = \frac{1}{2q_2 + q_1^2} \begin{bmatrix} 0 & 0 & -q_1 & -1 \\ 0 & 0 & 2q_2 & -q_1 \\ q_1 & -2q_2 & 0 & p_3 \\ 1 & q_1 & -p_3 & 0 \end{bmatrix},$$

which, in accordance with the theory, is non-degenerate. Observe that this expression actually does not depend on the choice of the leaf \mathcal{S}_ν in the foliation \mathcal{S} . The Hamiltonian projection $X_D = \Pi_D dH$ attains on every leaf \mathcal{S}_ν the form

$$\pi_{R_\nu} dh_{R_\nu} = \frac{1}{2q_2 + q_1^2} \left(\alpha_1 \frac{\partial}{\partial q_1} + \alpha_2 \frac{\partial}{\partial q_2} + \beta_2 \frac{\partial}{\partial p_2} + \beta_3 \frac{\partial}{\partial p_3} \right)$$

where the functions α_i, β_i are some rather complicated polynomial functions of coordinates and the parameters ν_i of the leaf \mathcal{S}_ν .

Let us now choose another distribution \mathcal{Z} , for which Π is also \mathcal{Z} -invariant. Since the operator Π has a very simple form, any pair of constant fields will span a distribution \mathcal{Z} that makes Π -invariant, since then the Vaisman condition (3) is trivially satisfied. Thus, let us take $\mathcal{Z} = Sp\left\{\frac{\partial}{\partial q_3} + \frac{\partial}{\partial p_3}, \frac{\partial}{\partial q_3} + \frac{\partial}{\partial p_1}\right\}$ (observe that this distribution is integrable). We have now to change the basis in \mathcal{Z} to

a new basis $\{Z_1, Z_2\}$ such that the condition $Z_i(\varphi_j) = \delta_{ij}$ is satisfied. A simple calculation yields

$$Z_1 = \frac{\partial}{\partial q_3} + \frac{1}{1-q_2} \left(-q_2 \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_3} \right), \quad Z_2 = \frac{1}{1-q_2} \left(\frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_3} \right).$$

Now, the general deformation (6) defined by the above distribution attains in the coordinates $\{(q_1, q_2, \varphi_1, \varphi_2, p_2, p_3)\}$ the form

$$\Pi_D = \frac{1}{q_2 - 1} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & q_2 - 1 & -q_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 - q_2 & 0 & 0 & 0 & p_3 - q_1 \\ 1 & q_1 & 0 & 0 & q_1 - p_3 & 0 \end{bmatrix}$$

and thus the restricted operator π_{R_ν} on the leaf \mathcal{S}_ν parametrized with the coordinates $\{(q_1, q_2, p_2, p_3)\}$ is

$$\pi_{R_\nu} = \frac{1}{q_2 - 1} \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & q_2 - 1 & -q_1 \\ 0 & 1 - q_2 & 0 & p_3 - q_1 \\ 1 & q_1 & q_1 - p_3 & 0 \end{bmatrix},$$

which is again non-degenerate. Again, this expression does not depend on the choice of the leaf \mathcal{S}_ν in the foliation \mathcal{S} .

In the end, consider yet another distribution that makes Π \mathcal{Z} -invariant, namely $\mathcal{Z} = Sp \left\{ \frac{\partial}{\partial q_3} + \frac{\partial}{\partial p_3}, \frac{\partial}{\partial p_1} \right\}$. The appropriate basis of \mathcal{Z} is given by

$$Z_1 = \frac{\partial}{\partial q_3} - q_2 \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_3}, \quad Z_2 = \frac{\partial}{\partial p_1}.$$

so that we have $Z_i(\varphi_j) = \delta_{ij}$. The general deformation (6) defined by the above distribution yields in the coordinates $\{(q_1, q_2, \varphi_1, \varphi_2, p_2, p_3)\}$

$$\Pi_D = \frac{1}{q_2 - 1} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & q_1 \\ 0 & 0 & 0 & 0 & -q_1 & 0 \end{bmatrix}$$

so that the restricted operator π_{R_ν} on the leaf \mathcal{S}_ν (again parametrized with the coordinates $\{(q_1, q_2, p_2, p_3)\}$) is

$$\pi_{R_\nu} = \frac{1}{q_2 - 1} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & q_1 \\ 0 & 0 & -q_1 & 0 \end{bmatrix},$$

and is degenerate this time.

Thus, given a foliation \mathcal{S} , by choosing different distributions \mathcal{Z} we can obtain several different Hamiltonian projections of our original Hamiltonian system, not only just the well known Dirac reduction.

5 Conclusions

In this article we formulated a comprehensive, geometric picture of what is known as Dirac reduction and Marsden-Ratiu reduction of a Poisson operator Π on a foliation \mathcal{S} not related with the symplectic foliation of Π . As a consequence, we obtain a geometric method of reducing of any Hamiltonian system to a Hamiltonian system on the foliation \mathcal{S} . Any such reduction depends merely on the choice of the distribution \mathcal{Z} along which the reduction takes place. Thus, we can reduce Hamiltonian systems on \mathcal{M} to Hamiltonian systems on \mathcal{S} in many non-equivalent ways. However, the procedure of finding appropriate distributions \mathcal{Z} (i.e. those that make Π \mathcal{Z} -invariant) is non-trivial and non-algorithmic.

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